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AN ISOMORPHIC VERSION OF THE BUSEMANN-PETTY PROBLEM FOR ARBITRARY MEASURES.

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ABSTRACT. The Busemann-Petty problem for an arbitrary measure μ with non-negative even continuous density in \mathbb{R}^n asks whether origin-symmetric convex bodies in \mathbb{R}^n with smaller $(n-1)$ -dimensional measure μ of all central hyperplane sections necessarily have smaller measure μ . It was shown in [Zv] that the answer to this problem is affirmative for $n \leq 4$ and negative for $n \geq 5$. In this paper we prove an isomorphic version of this result. Namely, if K, M are origin-symmetric convex bodies in \mathbb{R}^n such that $\mu(K \cap \xi^\perp) \leq \mu(M \cap \xi^\perp)$ for every $\xi \in \mathbb{S}^{n-1}$, then $\mu(K) \leq \sqrt{n} \mu(M)$. Here ξ^\perp is the central hyperplane perpendicular to ξ . We also study the above question with additional assumptions on the body K and present the complex version of the problem. In the special case where the measure μ is convex we show that \sqrt{n} can be replaced by cL_n , where L_n is the maximal isotropic constant. Note that, by a recent result of Klartag, $L_n \leq O(n^{1/4})$. Finally we prove a slicing inequality

$$\mu(K) \leq Cn^{1/4} \max_{\xi \in \mathbb{S}^{n-1}} \mu(K \cap \xi^\perp) \text{vol}_n(K)^{\frac{1}{n}}$$

for any convex even measure μ and any symmetric convex body K in \mathbb{R}^n , where C is an absolute constant. This inequality was recently proved in [K2] for arbitrary measures with continuous density, but with \sqrt{n} in place of $n^{1/4}$.

1. INTRODUCTION

Let f be a non-negative even continuous function on \mathbb{R}^n , and let μ be the measure in \mathbb{R}^n with density f , i.e. for every compact set $B \subset \mathbb{R}^n$

$$\mu(B) = \int_B f(x) dx.$$

This definition also applies to compact sets $B \subset \xi^\perp$, where $\xi \in \mathbb{S}^{n-1}$ and ξ^\perp is the central hyperplane orthogonal to ξ . The following problem was solved in [Zv].

Busemann-Petty problem for general measures (BPGM): Fix $n \geq 2$. Given two convex origin-symmetric bodies K and M in \mathbb{R}^n such that

$$\mu(K \cap \xi^\perp) \leq \mu(M \cap \xi^\perp)$$

for every $\xi \in \mathbb{S}^{n-1}$, does it follow that

$$\mu(K) \leq \mu(M)?$$

The BPGM problem is a triviality for $n = 2$ and strictly positive f , and the answer is “yes”, moreover $K \subseteq M$. It was proved in [Zv], that for every strictly positive density f the answer to BPGM is affirmative if $n \leq 4$ and negative if $n \geq 5$.

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The BPGM problem is a generalization of the original Busemann-Petty problem, posed in 1956 (see [BP]) and asking the same question for Lebesgue measure $\mu(K) = \text{vol}_n(K)$; see [Zh, GKS, Ga, K3] for the solution and historical details.

Since the answer to BPGM is negative in most dimensions, it is natural to consider the following question.

Isomorphic Busemann-Petty problem for general measures: *Does there exist a universal constant \mathcal{L} such that for any measure μ with continuous non-negative even density f and any two origin-symmetric convex bodies K and M in \mathbb{R}^n such that*

$$\mu(K \cap \xi^\perp) \leq \mu(M \cap \xi^\perp)$$

for every $\xi \in \mathbb{S}^{n-1}$, one necessarily has

$$\mu(K) \leq \mathcal{L} \mu(M)?$$

In Section 2 we give an answer to this question with a constant not depending on the measure or bodies, but dependent on the dimension, namely we show that one can take $\mathcal{L} = \sqrt{n}$. We do not know whether this constant is optimal for general measures, but we are able to improve the constant \sqrt{n} to $Cn^{1/4}$ for convex measures using the techniques of Ball [Ba1] and Bobkov [Bob]; see Section 4. We also (see the end of Section 2) provide better estimates under additional assumptions that K is a convex k -intersection body or K is the unit ball of a subspace of L_p . Finally, Section 3 is dedicated to the complex version of the isomorphic Busemann-Petty problem for arbitrary measures.

In the case of volume the isomorphic Busemann-Petty problem is closely related to the hyperplane problem of Bourgain [Bo1, Bo2, Bo3] which asks whether there exists an absolute constant C so that for any origin-symmetric convex body K in \mathbb{R}^n

$$\text{vol}_n(K)^{\frac{n-1}{n}} \leq C \max_{\xi \in \mathbb{S}^{n-1}} \text{vol}_{n-1}(K \cap \xi^\perp);$$

see [MP] or [BGVV] for the connection between these two problems. The hyperplane problem is still open, with the best-to-date estimate $C = O(n^{1/4})$ established by Klartag [Kl], who slightly improved the previous estimate of Bourgain [Bo3]. In Section 4, following recent results of Bobkov [Bob], we show that Klartag's result can be extended to all convex measures in the following form. There exists an absolute constant C so that for every even convex measure μ on \mathbb{R}^n and every origin-symmetric convex body K in \mathbb{R}^n

$$\mu(K) \leq Cn^{1/4} \max_{\xi \in \mathbb{S}^{n-1}} \mu(K \cap \xi^\perp) \text{vol}_n(K)^{\frac{1}{n}}.$$

Note that this inequality was proved in [K2] for arbitrary measures μ with even continuous density, but with the constant \sqrt{n} in place of $n^{1/4}$:

$$\mu(K) \leq \sqrt{n} \frac{n}{n-1} c_n \max_{\xi \in \mathbb{S}^{n-1}} \mu(K \cap \xi^\perp) \text{vol}_n(K)^{\frac{1}{n}}, \quad (1.1)$$

where $c_n = \text{vol}_n(B_2^n)^{\frac{n-1}{n}} / \text{vol}_{n-1}(B_2^{n-1}) < 1$ and B_2^n is the unit Euclidean ball in \mathbb{R}^n . Also, for some special classes of bodies, including unconditional bodies, k -intersection bodies, duals of bodies with bounded volume ratio, inequality (1.1) has been proved with an absolute constant in place of \sqrt{n} (see [K1, K4, K9]). Versions of (1.1) for lower dimensional sections can be found in [K5].

2. ISOMORPHIC BUSEMANN-PETTY PROBLEM WITH $\mathcal{L} = \sqrt{n}$

We need several definitions and facts. A closed bounded set K in \mathbb{R}^n is called a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points

different from the origin, the origin is an interior point of K , and the *Minkowski functional* of K defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}$$

is a continuous function on \mathbb{R}^n .

The *radial function* of a star body K is defined by

$$\rho_K(x) = \|x\|_K^{-1}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If $x \in \mathbb{S}^{n-1}$ then $\rho_K(x)$ is the radius of K in the direction of x .

If μ is a measure on K with even continuous density f , then

$$\mu(K) = \int_K f(x) dx = \int_{\mathbb{S}^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta) dr \right) d\theta. \quad (2.1)$$

Putting $f = 1$, one gets

$$\text{vol}_n(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K^n(\theta) d\theta = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \|\theta\|_K^{-n} d\theta. \quad (2.2)$$

The *spherical Radon transform* $\mathcal{R} : C(\mathbb{S}^{n-1}) \mapsto C(\mathbb{S}^{n-1})$ is a linear operator defined by

$$\mathcal{R}f(\xi) = \int_{\mathbb{S}^{n-1} \cap \xi^\perp} f(x) dx, \quad \xi \in \mathbb{S}^{n-1}$$

for every function $f \in C(\mathbb{S}^{n-1})$.

The polar formulas (2.1) and (2.2), applied to a hyperplane section of K , express volume of such a section in terms of the spherical Radon transform:

$$\begin{aligned} \mu(K \cap \xi^\perp) &= \int_{K \cap \xi^\perp} f = \int_{\mathbb{S}^{n-1} \cap \xi^\perp} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-2} f(r\theta) dr \right) d\theta \\ &= \mathcal{R} \left(\int_0^{\|\cdot\|_K^{-1}} r^{n-2} f(r \cdot) dr \right) (\xi). \end{aligned} \quad (2.3)$$

and

$$\text{vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{n-1} \int_{\mathbb{S}^{n-1} \cap \xi^\perp} \|\theta\|_K^{-n+1} d\theta = \frac{1}{n-1} \mathcal{R}(\|\cdot\|_K^{-n+1})(\xi). \quad (2.4)$$

The spherical Radon transform is self-dual (see [Gr, Lemma 1.3.3]), namely, for any functions $f, g \in C(\mathbb{S}^{n-1})$

$$\int_{\mathbb{S}^{n-1}} \mathcal{R}f(\xi) g(\xi) d\xi = \int_{\mathbb{S}^{n-1}} f(\xi) \mathcal{R}g(\xi) d\xi. \quad (2.5)$$

Using self-duality, one can extend the spherical Radon transform to measures. Let ν be a finite Borel measure on \mathbb{S}^{n-1} . We define the spherical Radon transform of ν as a functional $\mathcal{R}\nu$ on the space $C(\mathbb{S}^{n-1})$ acting by

$$(\mathcal{R}\nu, f) = (\nu, \mathcal{R}f) = \int_{\mathbb{S}^{n-1}} \mathcal{R}f(x) d\nu(x).$$

By Riesz's characterization of continuous linear functionals on the space $C(\mathbb{S}^{n-1})$, $\mathcal{R}\nu$ is also a finite Borel measure on \mathbb{S}^{n-1} . If ν has continuous density g , then by (2.5) the Radon transform of ν has density $\mathcal{R}g$.

The class of intersection bodies was introduced by Lutwak [Lu]. Let K, L be origin-symmetric star bodies in \mathbb{R}^n . We say that K is the intersection body of L if the radius of

K in every direction is equal to the $(n-1)$ -dimensional volume of the section of L by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in \mathbb{S}^{n-1}$,

$$\rho_K(\xi) = \|\xi\|_K^{-1} = |L \cap \xi^\perp|. \quad (2.6)$$

All bodies K that appear as intersection bodies of different star bodies form *the class of intersection bodies of star bodies*.

Note that the right-hand side of (2.6) can be written in terms of the spherical Radon transform using (2.4):

$$\|\xi\|_K^{-1} = \frac{1}{n-1} \int_{\mathbb{S}^{n-1} \cap \xi^\perp} \|\theta\|_L^{-n+1} d\theta = \frac{1}{n-1} \mathcal{R}(\|\cdot\|_L^{-n+1})(\xi).$$

It means that a star body K is the intersection body of a star body if and only if the function $\|\cdot\|_K^{-1}$ is the spherical Radon transform of a continuous positive function on \mathbb{S}^{n-1} . This allows to introduce a more general class of bodies. A star body L in \mathbb{R}^n is called an *intersection body* if there exists a finite Borel measure ν on the sphere \mathbb{S}^{n-1} so that $\|\cdot\|_L^{-1} = \mathcal{R}\nu$ as functionals on $C(\mathbb{S}^{n-1})$, i.e. for every continuous function f on \mathbb{S}^{n-1} ,

$$\int_{\mathbb{S}^{n-1}} \|x\|_L^{-1} f(x) dx = \int_{\mathbb{S}^{n-1}} \mathcal{R}f(x) d\nu(x). \quad (2.7)$$

Intersection bodies played an essential role in the solution of the Busemann-Petty problem; we refer the reader to [Ga, K3, KoY] for more information about intersection bodies. It was proved in [Zv] (Theorems 3, 4), that if K is an intersection body then the answer to the BPGM is affirmative for K and any convex symmetric body M , whose central sections have greater μ -measure than the corresponding sections of K .

We need the following simple fact; cf. [Zv, Lemma 1].

Lemma 1. *For any $\omega, a, b > 0$ and any measurable function $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ we have*

$$\frac{\omega}{a} \int_0^a t^{n-1} \alpha(t) dt - \omega \int_0^a t^{n-2} \alpha(t) dt \leq \frac{\omega}{a} \int_0^b t^{n-1} \alpha(t) dt - \omega \int_0^b t^{n-2} \alpha(t) dt, \quad (2.8)$$

provided all the integrals exist.

Proof. The desired inequality is equivalent to

$$a \int_a^b t^{n-2} \alpha(t) dt \leq \int_a^b t^{n-1} \alpha(t) dt.$$

□

Denote by

$$d_{BM}(K, L) = \inf\{d > 0 : \exists T \in GL(n) : K \subset TL \subset dK\}$$

the Banach-Mazur distance between two origin-symmetric convex bodies L and K in \mathbb{R}^n (see [MS, Section 3]), and let

$$d_I(K) = \min\{d_{BM}(K, L) : L \text{ is an intersection body in } \mathbb{R}^n\}.$$

Theorem 1. *For any measure μ with continuous, non-negative even density f on \mathbb{R}^n and any two convex origin-symmetric convex bodies $K, M \subset \mathbb{R}^n$ such that*

$$\mu(K \cap \xi^\perp) \leq \mu(M \cap \xi^\perp), \quad \forall \xi \in \mathbb{S}^{n-1} \quad (2.9)$$

we have

$$\mu(K) \leq d_I(K) \mu(M).$$

Proof.

First, we use the polar formula (2.3) to write the condition (2.9) in terms of the spherical Radon transform:

$$\mathcal{R} \left(\int_0^{\|\cdot\|_K^{-1}} r^{n-2} f(r \cdot) dr \right) (\xi) \leq \mathcal{R} \left(\int_0^{\|\cdot\|_M^{-1}} r^{n-2} f(r \cdot) dr \right) (\xi). \quad (2.10)$$

Next we consider an intersection body L such that $L \subset K \subset d_I(K)L$ (note that a linear image of an intersection body is again an intersection body; see for example [Ga, Theorem 8.1.6]) and integrate (2.10) over \mathbb{S}^{n-1} with respect to the measure ν corresponding to the intersection body L . Using (2.7) we get

$$\int_{\mathbb{S}^{n-1}} \|x\|_L^{-1} \int_0^{\|x\|_K^{-1}} t^{n-2} f(tx) dt dx \leq \int_{\mathbb{S}^{n-1}} \|x\|_L^{-1} \int_0^{\|x\|_M^{-1}} t^{n-2} f(tx) dt dx. \quad (2.11)$$

Now, we apply (2.8) with $\omega = \|x\|_L^{-1}$, $a = \|x\|_K^{-1}$, $b = \|x\|_M^{-1}$ and $\alpha(t) = f(tx)$ to get

$$\begin{aligned} \frac{\|x\|_L^{-1}}{\|x\|_K^{-1}} \int_0^{\|x\|_K^{-1}} t^{n-1} f(tx) dt - \|x\|_L^{-1} \int_0^{\|x\|_K^{-1}} t^{n-2} f(tx) dt \\ \leq \frac{\|x\|_L^{-1}}{\|x\|_K^{-1}} \int_0^{\|x\|_M^{-1}} t^{n-1} f(tx) dt - \|x\|_L^{-1} \int_0^{\|x\|_M^{-1}} t^{n-2} f(tx) dt. \end{aligned} \quad (2.12)$$

Integrating (2.12) over \mathbb{S}^{n-1} , adding it to (2.11) and using $L \subset K \subset d_I(K)L$ we get

$$\int_{\mathbb{S}^{n-1}} \frac{\|x\|_L^{-1}}{\|x\|_K^{-1}} \int_0^{\|x\|_K^{-1}} t^{n-1} f(tx) dt dx \leq \int_{\mathbb{S}^{n-1}} \frac{\|x\|_L^{-1}}{\|x\|_K^{-1}} \int_0^{\|x\|_M^{-1}} t^{n-1} f(tx) dt dx \quad (2.13)$$

and

$$\frac{1}{d_I(K)} \int_{\mathbb{S}^{n-1}} \int_0^{\|x\|_K^{-1}} t^{n-1} f(tx) dt dx \leq \int_{\mathbb{S}^{n-1}} \int_0^{\|x\|_M^{-1}} t^{n-1} f(tx) dt dx.$$

The result follows from (2.1). □

It is easy to see that the Euclidean ball B_2^n is an intersection body. By John's theorem (see, for example, [MS, Section 3] or [Ga, Theorem 4.2.12]), $d_{BM}(K, B_2^n) \leq \sqrt{n}$ for all convex origin-symmetric bodies $K \subset \mathbb{R}^n$. This immediately shows that $d_I(K) \leq \sqrt{n}$ for all convex origin-symmetric bodies $K \subset \mathbb{R}^n$. This fact together with Theorem 1 implies

Corollary 1. *For any measure μ with continuous non-negative even density on \mathbb{R}^n and any two convex origin-symmetric convex bodies $K, M \subset \mathbb{R}^n$ such that*

$$\mu(K \cap \xi^\perp) \leq \mu(M \cap \xi^\perp), \quad \forall \xi \in \mathbb{S}^{n-1} \quad (2.14)$$

we have

$$\mu(K) \leq \sqrt{n} \mu(M).$$

If the body K in Theorem 1 is an intersection body, the constant $\mathcal{L} = 1$ (see [Zv], Theorem 1); this is an analog of the well-known Lutwak's connection between intersection bodies and the Busemann-Petty problem. There are other special classes of bodies for which the constant \mathcal{L} does not depend on the dimension.

The classes of k -intersection bodies were introduced in [K6, K7]. For an integer k , $1 \leq k < n$ and star bodies D, L in \mathbb{R}^n , we say that D is the k -intersection body of L if for every $(n - k)$ -dimensional subspace H of \mathbb{R}^n ,

$$|D \cap H^\perp| = |L \cap H|,$$

where H^\perp is the k -dimensional subspace orthogonal to H . Taking the closure in the radial metric of the class of all D 's that appear as k -intersection bodies of star bodies, we define the class of k -intersection bodies (the original definition in [K6, K7] was different; the equivalence of definitions was proved by Milman [Mi]). These classes of bodies are important for generalizations of the Busemann-Petty problem; see [K3].

To estimate the Banach-Mazur distance from k -intersection bodies to intersection bodies, we use two results. The first was proved in [K8]; see also [K3, Theorem 4.11].

Proposition 1. *The unit ball of any finite dimensional subspace of L_q with $0 < q \leq 2$ is an intersection body.*

We also use a result from [KK]; see also [K3, Theorem 6.18].

Proposition 2. *For every $k \in \mathbb{N}$ and every $0 < q < 1$, there exists a constant $c(k, q)$ such that for every $n \in \mathbb{N}$, $n > k$ and every origin-symmetric convex k -intersection body D in \mathbb{R}^n there exists an n -dimensional subspace of $L_q([0, 1])$ whose unit ball L satisfies $L \subset D \subset c(k, q)L$.*

Corollary 2. *Let $k \in \mathbb{N}$. There exists a constant $C(k)$ such that for any $n > k$, any measure μ with continuous non-negative even density on \mathbb{R}^n , any convex k -intersection body K in \mathbb{R}^n and any origin-symmetric convex body $M \subset \mathbb{R}^n$, the inequalities*

$$\mu(K \cap \xi^\perp) \leq \mu(M \cap \xi^\perp), \quad \forall \xi \in \mathbb{S}^{n-1}$$

imply

$$\mu(K) \leq C(k) \mu(M).$$

Proof.

Let $q = 1/2$. Propositions 1 and 2 imply that $d_I(K) \leq c(k, 1/2) =: C(k)$. The result follows from Theorem 1. □

The constant \sqrt{n} in Corollary 1 can also be improved if K is the unit ball of a subspace of L_p , $p > 2$. For such K , by a result of Lewis [Le] (see also [SZ] for a different proof), we have $d_{BM}(K, B_2^n) \leq n^{1/2-1/p}$. Since B_2^n is an intersection body, Theorem 1 implies the following.

Corollary 3. *Let $p > 2$, let K be the unit ball of an n -dimensional subspace of L_p , and let μ be a measure with even continuous density on \mathbb{R}^n . Suppose that M is an origin-symmetric convex body in \mathbb{R}^n so that*

$$\mu(K \cap \xi^\perp) \leq \mu(M \cap \xi^\perp), \quad \forall \xi \in \mathbb{S}^{n-1}.$$

Then

$$\mu(K) \leq n^{1/2-1/p} \mu(M).$$

Remark. The statements of Theorem 1 and Corollaries 1, 2, 3 hold true if M is any star body.

3. THE COMPLEX CASE

Origin symmetric convex bodies in \mathbb{C}^n are the unit balls of norms on \mathbb{C}^n . We denote by $\|\cdot\|_K$ the norm corresponding to the body K :

$$K = \{z \in \mathbb{C}^n : \|z\|_K \leq 1\}.$$

In order to define volume, we identify \mathbb{C}^n with \mathbb{R}^{2n} using the standard mapping

$$\xi = (\xi_1, \dots, \xi_n) = (\xi_{11} + i\xi_{12}, \dots, \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}).$$

Since norms on \mathbb{C}^n satisfy the equality

$$\|\lambda z\| = |\lambda| \|z\|, \quad \forall z \in \mathbb{C}^n, \forall \lambda \in \mathbb{C},$$

origin symmetric complex convex bodies correspond to those origin symmetric convex bodies K in \mathbb{R}^{2n} that are invariant with respect to any coordinate-wise two-dimensional rotation, namely for each $\theta \in [0, 2\pi]$ and each $\xi = (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$

$$\|\xi\|_K = \|R_\theta(\xi_{11}, \xi_{12}), \dots, R_\theta(\xi_{n1}, \xi_{n2})\|_K, \quad (3.1)$$

where R_θ stands for the counterclockwise rotation of \mathbb{R}^2 by the angle θ with respect to the origin. We shall say that K is a *complex convex body in \mathbb{R}^{2n}* if K is a convex body and satisfies equations (3.1). Similarly, complex star bodies are R_θ -invariant star bodies in \mathbb{R}^{2n} .

For $\xi \in \mathbb{C}^n$, $|\xi| = 1$, denote by

$$H_\xi = \{z \in \mathbb{C}^n : (z, \xi) = \sum_{k=1}^n z_k \overline{\xi_k} = 0\}$$

the complex hyperplane through the origin, perpendicular to ξ . Under the standard mapping from \mathbb{C}^n to \mathbb{R}^{2n} the hyperplane H_ξ turns into a $(2n-2)$ -dimensional subspace of \mathbb{R}^{2n} .

Denote by $C_c(\mathbb{S}^{2n-1})$ the space of R_θ -invariant continuous functions, i.e. continuous real-valued functions f on the unit sphere \mathbb{S}^{2n-1} in \mathbb{R}^{2n} satisfying $f(\xi) = f(R_\theta(\xi))$ for all $\xi \in \mathbb{S}^{2n-1}$ and all $\theta \in [0, 2\pi]$. The *complex spherical Radon transform* is an operator $\mathcal{R}_c : C_c(\mathbb{S}^{2n-1}) \rightarrow C_c(\mathbb{S}^{2n-1})$ defined by

$$\mathcal{R}_c f(\xi) = \int_{\mathbb{S}^{2n-1} \cap H_\xi} f(x) dx.$$

We say that a finite Borel measure ν on \mathbb{S}^{2n-1} is R_θ -invariant if for any continuous function f on \mathbb{S}^{2n-1} and any $\theta \in [0, 2\pi]$,

$$\int_{\mathbb{S}^{2n-1}} f(x) d\nu(x) = \int_{\mathbb{S}^{2n-1}} f(R_\theta x) d\nu(x).$$

The complex spherical Radon transform of an R_θ -invariant measure ν is defined as a functional $\mathcal{R}_c \nu$ on the space $C_c(\mathbb{S}^{2n-1})$ acting by

$$(\mathcal{R}_c \nu, f) = \int_{\mathbb{S}^{2n-1}} \mathcal{R}_c f(x) d\nu(x).$$

Complex intersection bodies were introduced and studied in [KPZ]. An origin symmetric complex star body K in \mathbb{R}^{2n} is called a *complex intersection body* if there exists a finite Borel R_θ -invariant measure ν on \mathbb{S}^{2n-1} so that $\|\cdot\|_K^{-2}$ and $\mathcal{R}_c \nu$ are equal as functionals on $C_c(\mathbb{S}^{2n-1})$, i.e. for any $f \in C_c(\mathbb{S}^{2n-1})$

$$\int_{\mathbb{S}^{2n-1}} \|x\|_K^{-2} f(x) dx = \int_{\mathbb{S}^{2n-1}} \mathcal{R}_c f(\theta) d\nu(\theta). \quad (3.2)$$

It was proved in [KPZ] that an origin-symmetric complex star body K in \mathbb{R}^{2n} is a complex intersection body if and only if the function $\|\cdot\|_K^{-2}$ represents a positive definite distribution on \mathbb{R}^{2n} .

We need a polar formula for the measure of a complex star body K in \mathbb{R}^{2n} :

$$\mu(K) = \int_K f(x) dx = \int_{\mathbb{S}^{2n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{2n-1} f(r\theta) dr \right) d\theta. \quad (3.3)$$

For every $\xi \in \mathbb{S}^{2n-1}$,

$$\begin{aligned} \mu(K \cap H_\xi) &= \int_{K \cap H_\xi} f(x) dx \\ &= \int_{\mathbb{S}^{2n-1} \cap H_\xi} \left(\int_0^{\|\theta\|_K^{-1}} r^{2n-3} f(r\theta) dr \right) d\theta \\ &= \mathcal{R}_c \left(\int_0^{\|\cdot\|_K^{-1}} r^{2n-3} f(r\cdot) dr \right) (\xi), \end{aligned} \quad (3.4)$$

We use an elementary inequality, which is a modification of Lemma 1.

Lemma 2. *For any $\omega, a, b > 0$ and measurable function $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ we have*

$$\begin{aligned} &\frac{\omega^2}{a^2} \int_0^a t^{2n-1} \alpha(t) dt - \omega^2 \int_0^a t^{2n-3} \alpha(t) dt \\ &\leq \frac{\omega^2}{a^2} \int_0^b t^{2n-1} \alpha(t) dt - \omega^2 \int_0^b t^{2n-3} \alpha(t) dt, \end{aligned}$$

provided all the integrals exist.

Proof. By a simple rearrangement of integrals, the inequality follows from

$$a^2 \int_a^b t^{2n-3} \alpha(t) dt \leq \int_a^b t^{2n-1} \alpha(t) dt. \quad \square$$

Denote by

$$d_G(K, L) = \inf\{b/a : a, b > 0 \text{ and } aK \subset L \subset bK\}$$

the geometric distance between two origin-symmetric convex bodies L and K in \mathbb{R}^{2n} . For a complex star body K in \mathbb{R}^{2n} denote by

$$d_{IC}(K) = \min\{d_G(K, L) : L \text{ is a complex intersection body in } \mathbb{R}^{2n}\}.$$

Theorem 2. *Let K and M be origin symmetric complex star bodies in \mathbb{R}^{2n} , and let μ be a measure on \mathbb{R}^{2n} with even continuous non-negative density f . Suppose that for every $\xi \in \mathbb{S}^{2n-1}$*

$$\mu(K \cap H_\xi) \leq \mu(M \cap H_\xi). \quad (3.5)$$

Then

$$\mu(K) \leq (d_{IC}(K))^2 \mu(M).$$

Proof. Without loss of generality, we can assume that the density f is invariant with respect to rotations R_θ . In fact, we can consider the measure μ_c with the density

$$f_c(x) = \frac{1}{2\pi} \int_0^{2\pi} f(R_\theta(x)) d\theta,$$

then $\mu_c(K \cap H_\xi) = \mu(K \cap H_\xi)$ and $\mu_c(K) = \mu(K)$ for any complex star body K in \mathbb{R}^{2n} and any $\xi \in \mathbb{S}^{2n-1}$.

By (3.4), the condition (3.5) can be written as

$$\begin{aligned} & \mathcal{R}_c \left(\int_0^{\|\cdot\|_K^{-1}} r^{2n-3} f(r\cdot) dr \right) (\xi) \\ & \leq \mathcal{R}_c \left(\int_0^{\|\cdot\|_L^{-1}} r^{2n-3} f(r\cdot) dr \right) (\xi), \quad \forall \xi \in \mathbb{S}^{2n-1}. \end{aligned} \quad (3.6)$$

Let L be a complex intersection body in \mathbb{R}^{2n} such that $L \subset K \subset d_{IC}(K)L$. Integrate (3.6) over \mathbb{S}^{2n-1} with respect to the measure μ corresponding to the intersection body L by (3.2). By (3.2)

$$\begin{aligned} & \int_{\mathbb{S}^{2n-1}} \|x\|_L^{-2} \int_0^{\|x\|_K^{-1}} t^{2n-3} f(tx) dt dx \\ & \leq \int_{\mathbb{S}^{2n-1}} \|x\|_L^{-2} \int_0^{\|x\|_M^{-1}} t^{2n-3} f(tx) dt dx. \end{aligned} \quad (3.7)$$

By Lemma 2 with $\omega = \|x\|_L^{-1}$, $a = \|x\|_K^{-1}$, $b = \|x\|_M^{-1}$ and $\alpha(t) = f(tx)$,

$$\begin{aligned} & \frac{\|x\|_L^{-2}}{\|x\|_K^{-2}} \int_0^{\|x\|_K^{-1}} t^{2n-1} f(tx) dt - \|x\|_L^{-2} \int_0^{\|x\|_K^{-1}} t^{2n-3} f(tx) dt \\ & \leq \frac{\|x\|_L^{-2}}{\|x\|_K^{-2}} \int_0^{\|x\|_M^{-1}} t^{2n-1} f(tx) dt - \|x\|_L^{-2} \int_0^{\|x\|_M^{-1}} t^{2n-3} f(tx) dt. \end{aligned} \quad (3.8)$$

Integrating (3.8) over \mathbb{S}^{2n-1} and adding it to (3.7) we get

$$\int_{\mathbb{S}^{2n-1}} \frac{\|x\|_L^{-2}}{\|x\|_K^{-2}} \int_0^{\|x\|_K^{-1}} t^{2n-1} f(tx) dt dx \leq \int_{\mathbb{S}^{2n-1}} \frac{\|x\|_L^{-2}}{\|x\|_K^{-2}} \int_0^{\|x\|_M^{-1}} t^{2n-1} f(tx) dt dx. \quad (3.9)$$

Since $L \subset K \subset d_{IC}(K)L$, the latter inequality gives

$$\frac{1}{(d_{IC}(K))^2} \int_{\mathbb{S}^{2n-1}} \int_0^{\|x\|_K^{-1}} t^{2n-1} f(tx) dt dx \leq \int_{\mathbb{S}^{2n-1}} \int_0^{\|x\|_M^{-1}} t^{2n-1} f(tx) dt dx.$$

The result follows from (3.3). \square

Corollary 4. *Suppose that K and M are origin-symmetric complex convex bodies in \mathbb{R}^{2n} and μ is an arbitrary measure on \mathbb{R}^{2n} with even continuous density so that*

$$\mu(K \cap H_\xi) \leq \mu(M \cap H_\xi), \quad \forall \xi \in \mathbb{S}^{2n-1},$$

then

$$\mu(K) \leq 2n \mu(M).$$

Proof. By John's theorem (see, for example, [MS, Section 3] or [Ga, Theorem 4.2.12]), there exists an origin symmetric ellipsoid \mathcal{E} such that

$$\frac{1}{\sqrt{2n}}\mathcal{E} \subset K \subset \mathcal{E}.$$

Construct a new body \mathcal{E}_c by

$$\|x\|_{\mathcal{E}_c}^{-2} = \frac{1}{2\pi} \int_0^{2\pi} \|R_\theta x\|_{\mathcal{E}}^{-2} d\theta.$$

Clearly, \mathcal{E}_c is R_θ -invariant, so it is a complex star body. For every $\theta \in [0, 2\pi]$ the distribution $\|R_\theta x\|_{\mathcal{E}}^{-2}$ is positive definite, because of the connection between the Fourier transform and linear transformations. So $\|x\|_{\mathcal{E}_c}^{-2}$ is also a positive definite distribution, and, by [KPZ, Theorem 4.1], \mathcal{E}_c is a complex intersection body. Since $\frac{1}{\sqrt{2n}}\mathcal{E} \subset K \subset \mathcal{E}$ and K is R_θ -invariant as a complex convex body, we have

$$\frac{1}{\sqrt{2n}}R_\theta\mathcal{E} \subset K \subset R_\theta\mathcal{E}, \quad \forall \theta \in [0, 2\pi],$$

so

$$\frac{1}{\sqrt{2n}}\mathcal{E}_c \subset K \subset \mathcal{E}_c.$$

Therefore, $d_{IC}(K) \leq \sqrt{2n}$, and the result follows from Theorem 2. \square

4. THE CASE OF CONVEX MEASURES

Following works of Borell [Bor1, Bor2], we define the classes of s -concave measures. Let $s \in [-\infty, 1]$. A measure μ on \mathbb{R}^n is called s -concave if for any compact $A, B \subset \mathbb{R}^n$, with $\mu(A)\mu(B) > 0$ and $0 < \lambda < 1$, we have

$$\mu(\lambda A + (1 - \lambda)B) \geq (\lambda\mu(A)^s + (1 - \lambda)\mu(B)^s)^{1/s}.$$

The case where $s = 0$ corresponds to *log-concave* measures

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{(1-\lambda)},$$

and the case $s = -\infty$ corresponds to *convex measures*:

$$\mu(\lambda A + (1 - \lambda)B) \geq \min\{\mu(A), \mu(B)\}.$$

We also note that the class of convex measures is the largest class in this group in the sense that it contains all other s -concave measures. Due to this fact, we concentrate our attention on convex measures.

Borell [Bor1, Bor2] has shown that a measure μ on \mathbb{R}^n whose support is not contained in any affine hyperplane is a convex measure if and only if it is absolutely continuous with respect to Lebesgue measure, and its density f is a $-1/n$ -concave function on its support, i.e.

$$f(\lambda x + (1 - \lambda)y) \geq (\lambda f(x)^{-1/n} + (1 - \lambda)f(y)^{-1/n})^{-n}$$

for all $x, y : f(x), f(y) > 0$ and $\lambda \in [0, 1]$. Note that it follows from the latter definition that if $f(x)$ is a $-1/n$ -concave function then $f^{-1/n}$ is a convex function on its support.

We need the following theorem of Bobkov ([Bob], Theorem 2.1) which is a generalization of the previous result of Ball [Ba1] (we also refer to [CFPP] for a simpler proof).

Theorem 3. *Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be an even $-1/n$ -concave function on its support, satisfying $0 < \int_{\mathbb{R}^n} f < \infty$. Then the map*

$$x \rightarrow \left(\int_0^\infty f(rx) r^{n-2} dr \right)^{-\frac{1}{n-1}}$$

defines a norm on \mathbb{R}^n .

An immediate consequence of the Ball-Bobkov theorem is a very useful technique of connecting a convex measure of one convex body with volume of another convex body. This technique allows to generalize a number of classical results on Lebesgue measure to the case of convex measures (see [Ba1], [Bob], [KYZ] and [CFPP]). Namely, given the density f of a convex measure μ and a convex symmetric body K we define a body K_f by

$$\|x\|_{K_f} = \left((n-1) \int_0^\infty (1_K f)(rx) r^{n-2} dr \right)^{-\frac{1}{n-1}},$$

where 1_K is the indicator function of K . Theorem 3 guarantees that K_f is convex. Moreover, by (2.4)

$$\begin{aligned} \text{vol}_{n-1}(K_f \cap \xi^\perp) &= \frac{1}{n-1} \int_{\mathbb{S}^{n-1} \cap \xi^\perp} \|x\|_{K_f}^{-(n-1)} dx \\ &= \int_{\mathbb{S}^{n-1} \cap \xi^\perp} \int_0^\infty (1_K f)(rx) r^{n-2} dr \, dx = \mu(K \cap \xi^\perp). \end{aligned} \quad (4.1)$$

Our next goal is to estimate $\text{vol}_n(K_f)$. We start with a lemma on the behavior of $-1/n$ -concave functions, the proof of which may be found in [Kl, Lemma 2.4] and [Bob, Lemma 4.2].

Lemma 3. *Let $n \geq 1$ be an integer, and let $g : [0, \infty) \rightarrow [0, \infty)$ be a $-1/n$ -concave, non-increasing function with $g(0) = 1$, $0 < \int_0^\infty g(t) t^{n-2} dt < \infty$. Then*

$$c_1 \leq \frac{\int_0^\infty t^{n-1} g(t) dt}{\left(\int_0^\infty t^{n-2} g(t) dt \right)^{\frac{n}{n-1}}} \leq c_2,$$

where $c_1, c_2 > 0$ are universal constants.

Remark: We need the $-1/n$ -concavity assumption only to prove the right-hand side inequality in the above lemma. The left-hand side does not require this assumption, but does require $g \leq e^n$.

Now assume that $f(0) = 1$, f is even and $-1/n$ -concave, then $f(tx)$ is non-increasing for $t \geq 0$ and the function $g(t) = (1_K f)(tx)$ satisfies the conditions of Lemma 3. By (2.2)

$$\text{vol}_n(K_f) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \|x\|_{K_f}^{-n} dx = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \left((n-1) \int_0^\infty (1_K f)(rx) r^{n-2} dr \right)^{\frac{n}{n-1}} dx$$

and applying Lemma 3 we get

$$c_1 \mu(K) \leq \text{vol}_n(K_f) \leq c_2 \mu(K), \quad (4.2)$$

where $c_1, c_2 > 0$ are universal constants (and the right-hand side inequality does not require $-1/n$ -concavity, but does require boundness of f).

We refer to [MP] for the definition of the isotropic constant L_K of a convex body K . It was proved in [MP] that if $L_n = \max\{L_K : K \text{ is a convex symmetric body in } \mathbb{R}^n\}$ then from

$$\text{vol}_{n-1}(K \cap \xi^\perp) \leq \text{vol}_{n-1}(M \cap \xi^\perp)$$

we get $\text{vol}_n(K) \leq cL_n \text{vol}_n(M)$. Applying this fact to bodies K_f and M_f we immediately get the following theorem.

Theorem 4. *For any measure μ with continuous, non-negative even $-1/n$ -concave density f on \mathbb{R}^n and any two convex origin-symmetric bodies $K, M \subset \mathbb{R}^n$ such that*

$$\mu(K \cap \xi^\perp) \leq \mu(M \cap \xi^\perp), \quad \forall \xi \in \mathbb{S}^{n-1} \quad (4.3)$$

we have

$$\mu(K) \leq cL_n \mu(M).$$

Remark: We note that the assumption $f(0) = 1$ is not necessary in the above theorem due to the fact that the theorem does not change when μ is multiplied by a constant. It was proved by Bourgain that $L_n \leq cn^{1/4} \log(n+1)$ and the $\log(n+1)$ factor was after removed by Klartag [Kl], which implies the following corollary.

Corollary 5. *For any convex measure μ with continuous, non-negative even density f on \mathbb{R}^n and any two convex origin-symmetric bodies $K, M \subset \mathbb{R}^n$ such that*

$$\mu(K \cap \xi^\perp) \leq \mu(M \cap \xi^\perp), \quad \forall \xi \in \mathbb{S}^{n-1} \quad (4.4)$$

we have

$$\mu(K) \leq cn^{1/4} \mu(M).$$

It was also proved in [MP] that for any convex origin-symmetric body $K \subset \mathbb{R}^n$

$$\max_{\xi \in \mathbb{S}^{n-1}} \text{vol}_{n-1}(K \cap \xi^\perp) \geq \frac{c}{L_K} \text{vol}_n(K)^{\frac{n-1}{n}}.$$

which gives (applying the latter inequality to K_f)

$$\frac{1}{f(0)} \max_{\xi \in \mathbb{S}^{n-1}} \mu(K \cap \xi^\perp) \geq \frac{c}{L_K} \left(\frac{\mu(K)}{f(0)} \right)^{\frac{n-1}{n}},$$

which implies

Corollary 6. *For any convex measure μ with continuous, non-negative even density f on \mathbb{R}^n and any convex origin-symmetric body $K \subset \mathbb{R}^n$ we have*

$$\max_{\xi \in \mathbb{S}^{n-1}} \mu(K \cap \xi^\perp) \geq \frac{c}{L_K} \mu(K)^{\frac{n-1}{n}} f(0)^{\frac{1}{n}}.$$

Using convexity of μ we get that $\frac{\mu(K)}{f(0)} \leq \text{vol}_n(K)$, which proves the following hyperplane inequality for convex measures.

Corollary 7. *For any convex measure μ with continuous, non-negative even density f on \mathbb{R}^n and any convex origin-symmetric body $K \subset \mathbb{R}^n$ we have*

$$\max_{\xi \in \mathbb{S}^{n-1}} \mu(K \cap \xi^\perp) \geq \frac{c}{L_K} \mu(K) \text{vol}_n(K)^{-\frac{1}{n}},$$

and thus

$$\mu(K) \leq Cn^{1/4} \max_{\xi \in \mathbb{S}^{n-1}} \mu(K \cap \xi^\perp) \text{vol}_n(K)^{\frac{1}{n}}.$$

We would like to note that Corollary 6 was essentially proved by Bobkov [Bob, Theorem 4.1]. Our goal is a generalization of the hyperplane inequality to the case of most general measures with positive even and continuous density. We note that Corollary 6 is false in the case of general measures. Indeed, consider $f(x) = 1/(1 + |x|^p)$ for some $p \in (0, n)$, then $f(x)$ is radial decreasing and $f(0) = 1$ is still the maximum for f on \mathbb{R}^n . Let $K = tB_2^n$ for t large enough, then using (2.1) we get

$$\mu(tB_2^n) = \frac{1}{|\mathbb{S}^{n-1}|} \int_0^t \frac{r^{n-1} dr}{1 + r^p} \geq \frac{ct^{n-p}}{(n-p)|\mathbb{S}^{n-1}|}$$

and

$$\mu(tB_2^{n-1}) \leq \frac{t^{n-p-1}}{(n-p-1)|\mathbb{S}^{n-2}|}.$$

Thus for Corollary 6 to be correct we must have for all large t

$$t^{n-p-1} \geq c_n t^{(n-p)\frac{n-1}{n}}$$

or

$$n - p - 1 \geq (n - p) \frac{n - 1}{n} \quad \text{and} \quad p \leq 0,$$

which gives a contradiction.

We finish this note with an observation related to the hyperplane inequality for measures.

Lemma 4. *For any measure μ with continuous, non-negative even density f on \mathbb{R}^n consider a symmetric star-shaped body $K \subset \mathbb{R}^n$ such that $\mu(K \cap \xi^\perp) = \mu(K \cap \theta^\perp)$ for all $\xi, \theta \in \mathbb{S}^{n-1}$, then*

$$\mu(K) \leq C \mu(K \cap \xi^\perp) \text{vol}_n(K)^{\frac{1}{n}}, \quad \forall \xi \in \mathbb{S}^{n-1}.$$

Proof. Assume $\mu(K \cap \xi^\perp) = \Lambda$, then applying (2.3) we get

$$\mathcal{R} \left(\int_0^{\|\cdot\|_K^{-1}} r^{n-2} f(r \cdot) dr \right) (\xi) = \Lambda, \quad (4.5)$$

and applying the Funk-Minkowski uniqueness theorem for the spherical Radon transform (see for example [K3]) we get

$$\int_0^{\|\xi\|_K^{-1}} r^{n-2} f(r\xi) dr = \frac{\Lambda}{\text{vol}_{n-2}(\mathbb{S}^{n-2})}, \quad \forall \xi \in \mathbb{S}^{n-1}.$$

We also note that

$$\int_0^{\|\xi\|_K^{-1}} r^{n-1} f(r\xi) dr \leq \|\xi\|_K^{-1} \int_0^{\|\xi\|_K^{-1}} r^{n-2} f(r\xi) dr = \frac{\Lambda \|\xi\|_K^{-1}}{\text{vol}_{n-2}(\mathbb{S}^{n-2})}.$$

Finally, integrating the above inequality over $\xi \in \mathbb{S}^{n-1}$ and applying (2.1)

$$\begin{aligned} \mu(K) &\leq \frac{\Lambda}{\text{vol}_{n-2}(\mathbb{S}^{n-2})} \int_{\mathbb{S}^{n-1}} \|\xi\|_K^{-1} d\xi \\ &\leq \frac{\Lambda \text{vol}_{n-1}(\mathbb{S}^{n-1})^{\frac{n-1}{n}}}{\text{vol}_{n-2}(\mathbb{S}^{n-2})} \left(\int_{\mathbb{S}^{n-1}} \|\xi\|_K^{-n} d\xi \right)^{1/n} \\ &\leq C \Lambda \text{vol}_n(K)^{1/n}. \end{aligned} \quad (4.6)$$

□

Remark: We note that the body K in Lemma 4 exists for all $\Lambda > 0$ such that

$$\Lambda \leq \text{vol}_{n-2}(\mathbb{S}^{n-2}) \min_{\xi \in \mathbb{S}^{n-1}} \int_0^\infty r^{n-2} f(r\xi) dr.$$

This follows from (4.5), properties of the spherical Radon transform and the fact that $f(x) \geq 0$ (see Corollary 1 in [Zv]). Clearly, K is not necessarily a convex body. It seems to be quite difficult to find a sufficient condition on f for K to be convex. For any rotation invariant f we get that K is a dilate of the Euclidean ball.

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